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AN APPROXIMATION TECHNIQUE FOR NATURAL CONVECTION IN A BOUNDARY LAYER

J. BRINDLEY*

Department of Mathematics, University of Leeds

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Abstract—An approximation technique, widely used by Meksyn for finding solutions in terms of asymptotic expansions to problems of flow in boundary layers, is extended to free convection flows, and applied to the classical problem of free convection at a uniformly heated vertical wall in fluid otherwise at rest, and to the rather less known problems of combined free and forced convection at a vertical wall in a fluid having a vertical velocity at a large distance from the wall. It is found that, in the problems considered, the first three terms of the asymptotic series provide a good approximation to known results, and since in this case the essential computational problem is that of finding the least root of a quartic equation in which the Prandtl number appears as a parameter, the method is a good deal more easy to use and of more general application than those used by previous workers on these problems. Other problems of free convection, or combined free and forced convection, in which similarity transformations may be used are at once amenable to the same technique.

NOMENCLATURE

x, y,	co-ordinates along and normal to the wall;
ξ, η,	similarity co-ordinates;
u, v,	vertical and horizontal velocity components;
T,	absolute temperature;
ψ,	stream function;
$F(\eta)$,	non-dimensionalized temperature in similarity co-ordinates;
$f(\eta)$,	non-dimensionalized stream function in similarity co-ordinates;
g,	acceleration due to gravity;
ν,	kinematic viscosity;
β,	coefficient of thermal expansion;
k,	thermal conductivity;
σ,	Prandtl number;
$T_0(x)$,	temperature of ambient fluid;
$T_1(x)$,	temperature of wall;
$\Delta T(x)$,	temperature difference between wall and ambient fluid;
$U_0(x)$,	mainstream velocity;
Nu,	Nusselt number;
Gr,	Grashof number;
Re,	Reynolds number.

* Lecturer in Applied Mathematics.

INTRODUCTION

THE problem of free convection at a heated vertical wall in a fluid otherwise at rest has stimulated a good deal of work, both experimental and theoretical, since the early papers of Schmidt and Beckmann [1]. Extensive discussion of the wide physical importance of this type of problem appears elsewhere, for example Ostrach [2], and is therefore not reproduced here. Notable contributions towards a theoretical understanding of the problem have come from Saunders [3], Schuh [4], Ostrach [2], Sparrow and Gregg [5] and Reeves and Kippenham [6], whilst Lorenz [7], Eckert and Soehngen [8] and Scherberg [9] are amongst those who have carried out detailed experimental work. All analytical approaches have been based on similarity transformations of the equations of motion and energy, as have most purely numerical approaches, and the process of solution has usually been rather tedious.

The problem of convection in a boundary layer at a heated vertical wall in a fluid which has a vertical velocity at a large distance from the wall has received much less attention however. The most important theoretical contribution to this problem of combined free and forced convection has come from Sparrow, Eichhorn and Gregg [10], and we follow their precedent here in seeking similarity solutions of the equations of motion and energy. The existence of similarity solutions depends on a relationship between the variation of wall temperature and of free stream velocity, thus limiting the range of problems covered by the analysis, and in fact excluding the case of uniform wall temperature and uniform stream velocity. This particular case has, however, received some attention in the past, notably from Tanaev [11], Acrivos [12] and Sparrow and Gregg [13]. In this paper a method of asymptotic expansion, widely developed by Meksyn [14], and fully described and discussed by him, is used to find approximate solutions to two problems of combined free and forced convection, and to the classical problem of pure free convection at a uniformly heated vertical wall. This latter problem provides a useful vehicle for the development of the technique, and also for the assessment of its accuracy.

1. THE EQUATIONS OF MOTION AND ENERGY

The equations defining the problem are the equation of motion in the x-direction, x being measured vertically along the wall from the bottom edge, which with the usual boundary layer approximation is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} + g\beta(T - T_o) \tag{1.1}$$

where $T_o = T_o(x)$ is the temperature of the fluid at height x at a large distance from the wall; the equation of heat transfer, again with the usual boundary-layer simplifications

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = k\frac{\partial^2 T}{\partial y^2} \tag{1.2}$$

and the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1.3}$$

As is customary in boundary-layer theory we replace the pressure gradient in (1.1) by its value outside the layer, and neglect the variation of ρ. Thus we have

$$\frac{1}{\rho_o}\frac{\partial \rho_o}{\partial x} = U_o\frac{\partial U_o}{\partial x}$$

and we can rewrite (1.1) in the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_o\frac{\partial U_o}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} + g\beta(T - T_o). \tag{1.4}$$

We wish to examine the possibility of finding similarity solutions to (1.2), (1.3) and (1.4), and is is convenient to introduce a stream function $\psi(x, y)$, such that

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \tag{1.5}$$

and then make a transformation of variables of the form

$$\xi = x, \quad \eta = \frac{C_1 y}{G(x)}, \quad f(\xi, \eta) = \frac{\psi(x, y)}{\nu C_2 \Delta T(x) H(x)},$$
$$F(\xi, \eta) = \frac{T(x, y) - T_o(x)}{\Delta T(x)} \tag{1.6}$$

where $G(x)$ and $H(x)$ are functions of x only, and $T(\Delta x)$ is the temperature difference between the heated surface and the fluid at a large distance from the surface at the given value of x. The form of the functions $G(x)$, $H(x)$, $\Delta T(x)$ in order that equations (1.2)–(1.4) should become ordinary differential equations in η is readily found (see, for example, Yang [15]), and in particular the only functions $\Delta T(x)$ admitting of a similarity solution are of the form $\Delta T(x) = Ax^m$, and $Be^{\epsilon x}$. In the case $\Delta T(x) \propto x^m$ we must have $U_o \propto x^n$, where $2n - 1 = m$, and in the case $\Delta T(x) \propto e^{\epsilon x}$ we must have $U_o \propto e^{\frac{1}{2}\epsilon x}$. Thus the existence of similarity solutions of the equations in the case of combined forced and free convection is dependent on a quite intimate relationship between the forms of variation of the mainstream velocity and the temperature difference, and the number of configurations which can be investigated by this method is thereby limited. However, a number of interesting cases are accessible, and some of these we consider later.

2. THE CONSTANT WALL TEMPERATURE PROBLEM WITH NO MAINSTREAM VELOCITY

In order to make a comparison with known results it is convenient to consider first the particular case in which $\Delta T(x)$ is constant and $U_o(x)$ is zero. As mentioned earlier, this problem has been extensively studied, both theoretically and experimentally, and so provides a useful test of the accuracy of the method.

Following Schmidt and Beckmann [1] we write

$$\eta = \left(\frac{g\beta\Delta T}{4\nu^2}\right)^{1/4} \frac{y}{x^{1/4}} = A\frac{y}{x^{1/4}}$$

$$\psi(x, y) = 4\nu\, Ax^{3/4}f(\eta). \quad T(x, y) - T_o = \Delta TF(\eta) \tag{2.1}$$

and (1.1) and (1.2) simplify to

$$f''' + 3ff'' - 2f'^2 + F = 0 \tag{2.2}$$

$$F'' + 3\sigma fF' = 0. \tag{2.3}$$

In order to obtain solutions of (2.2) and (2.3) we make use of the fact that f'' and F' are large only in a narrow region close to the wall; we can therefore seek asymptotic solutions of the equations in order to employ the boundary conditions at infinity. To use the boundary conditions at $\eta = 0$ we first assume solutions for f and F in the form

$$f = \sum_{r=0}^{\infty} a_r\eta^r, \quad F = \sum_{r=0}^{\infty} \alpha_r\eta^r. \tag{2.4}$$

Then the boundary conditions $f = f' = 0$, $F = 1$ at $\eta = 0$ show that

$$a_o = a_1 = 0, \quad \alpha_o = 1. \tag{2.5}$$

Hence

$$f = \sum_{r=2}^{\infty} a_r\eta^r, \quad F = 1 + \sum_{1}^{\infty} \alpha_r\eta^r. \tag{2.6}$$

If we substitute these expressions (2.6) in (2.2) and (2.3) we find that

$$\sum_3^\infty a_r r(r-1)(r-2)\eta^{r-3} + 3\sum_2^\infty a_r\eta^r \sum_2^\infty a_r r(r-1)\eta^{r-2} - 2\left(\sum_2^\infty a_r r\eta^{r-1}\right)^2 + 1 + \sum_1^\infty \alpha_r\eta^r = 0 \tag{2.7}$$

$$\sum_2^\infty \alpha_r r(r-1)\eta^{r-2} + 3\sigma\sum_2^\infty a_r\eta^r \sum_2^\infty \alpha_r r\eta^{r-1} = 0 \tag{2.8}$$

whence, equating to zero the coefficients of corresponding powers of η

$$\eta^0 \begin{cases} 6a_3 + 1 = 0, & a_3 = -\dfrac{1}{6}. \\ 2\alpha_2 = 0. & \alpha_2 = 0. \end{cases}$$

$$\eta^1 \begin{cases} 24a_4 + \alpha_1 = 0, & \alpha_1 = -24a_4. \\ 6\alpha_3 = 0, & \alpha_3 = 0. \end{cases}$$

$$\eta^2 \begin{cases} 60a_5 + 6a_2^2 - 8a_2^2 + \alpha_2 = 0, & a_5 = \dfrac{1}{30}a_2^2. \\ 12\alpha_4 + 3\sigma a_2\alpha_1 = 0, & \alpha_4 = -\dfrac{1}{4}\sigma a_2\alpha_1 \\ & = 6\sigma a_2a_4. \end{cases}$$

$$\eta^3 \begin{cases} 120a_6 + \alpha_3 = 0, & a_6 = 0. \\ 20\alpha_5 + 3\sigma a_3\alpha_1 = 0, & \alpha_5 = -\dfrac{3}{5}\sigma a_4, \text{ etc.} \end{cases}$$

So we can write

$$\left.\begin{aligned} f &= a_2\eta^2 - \frac{1}{6}\eta^3 + a_4\eta^4 + \frac{1}{30}a_2^2\eta^5 \\ &\quad - a_2a_4\left(\frac{1}{21} + \frac{\sigma}{35}\right)\eta^7 + \ldots \\ F &= 1 - 24a_4\eta + 6\sigma a_2a_4\eta^4 - \frac{3}{5}\sigma a_4\eta^5 \\ &\quad + \frac{12}{5}\sigma a_4^2\eta^6 \ldots \end{aligned}\right\} \tag{2.9}$$

where all the coefficients may be expressed in terms of the two constants a_2 and a_4 which are found by using the boundary conditions at infinity. We substitute in (2.2) and (2.3) the above expressions for f, f' and F in order to obtain linear equations for f'' and F'; writing ω and ρ for the series expressions for f and F in (2.9) we have

$$f''' + 3\omega f'' = 2\omega'^2 - \rho \tag{2.10}$$

$$F'' + 3\sigma\omega F' = 0 \tag{2.11}$$

which may be integrated at once to give

$$f'' = \exp[-3\textstyle\int\omega(\eta)]\left[\int_0^\eta \exp[3\int\omega(\eta)](2\omega'^2 - \rho)\,d\eta + C\right]$$

$$F' = D\exp[-3\sigma\textstyle\int\omega(\eta)] \tag{2.12}$$

where C and D are constants.

To complete the solution we have to evaluate the integrals

$$f' = \int \exp[-3\int\omega(\eta)]\,\phi(\eta)\,d\eta,$$
$$F = 1 + D\int \exp[-3\sigma\int\omega(\eta)]\,d\eta \quad (2.13)$$

where

$$\phi(\eta) = \int_0^\eta \exp[3\int\omega(\eta)](2\omega'^2 - \rho)\,d\eta + C. \quad (2.14)$$

Integrating (2.9) term by term we have

$$3\int\omega(\eta)\,d\eta = a_2\eta^3 - \frac{1}{8}\eta^4 + \frac{3}{5}a_4\eta^5 + \frac{1}{60}a_2^2\eta^6 - \frac{3}{8}a_2a_4\left(\frac{1}{21}+\frac{\sigma}{35}\right)\eta^8 + \ldots = \tau, \text{ say} \quad (2.15)$$

and we can now replace η by τ as the independent variable in the expression (2.13) for f', viz.

$$f' = \int e^{-\tau}\phi(\eta)\,d\eta = \int e^{-\tau}\phi(\eta)\frac{d\eta}{d\tau}\,d\tau. \quad (2.16)$$

Since $\tau(\eta)$ starts with η^3, when we express η in terms of τ we can write

$$\eta = \sum_{m=1}^{\infty}\frac{A_m}{m+1}\tau^{1/3\,(m+1)} \quad (2.17)$$

which is valid for sufficiently small values of τ.

Thus

$$d\eta = \sum_{m=0}^{\infty}\frac{1}{3}A_m\,\tau^{1/3\,(m-2)}\,d\tau \quad (2.18)$$

from which it follows that

$$\oint_1 \frac{d\eta}{\tau^{1/3\,(m+1)}} = \frac{A_m}{3}\oint_3\frac{d\tau}{\tau} = 2\pi i A_m \quad (2.19)$$

the first integral being taken once round a circuit containing $\eta = 0$ in the η-plane, and the second integral being taken three times round a circuit containing $\tau = 0$ in the τ-plane.

It follows at once that A_m is the coefficient of η^{-1} in the expansion of $\tau^{-1/3\,(m+1)}$ in ascending powers of η; if we write $\int\omega(\eta)\,d\eta = \eta^3\sum_{n=0}^{\infty}c_n\eta^n = \tau$, we have

$$\tau^{-1/3\,(m+1)} = \eta^{-(m+1)}(c_0 + c_1\eta + \ldots)^{-1/3\,(m+1)} \quad (2.20)$$

whence A_m is the coefficient of η^m in the expression $(c_0 + c_1\eta + \ldots)^{-1/3(m+1)}$.

In fact we have

$$\tau = \eta^3\left(a_2 - \frac{1}{8}\eta + \frac{3}{5}a_4\eta^2 + \frac{1}{60}a_2^2\eta^3 + \ldots\right) \quad (2.15)$$

so A_m is the coefficient of η^m in the expression

$$\left|a_2 - \frac{1}{8}\eta + \frac{3}{5}a_4\eta^2 + \frac{1}{60}a_2^2\eta^3 - \frac{3}{8}a_2a_4\left(\frac{1}{21}+\frac{\sigma}{35}\right)\eta^5 + \ldots\right|^{-1/3\,(m+1)}. \quad (2.21)$$

We consider next the integral solution for $f'(\eta)$, given by

$$f' = \int \exp[-3\int\omega(\eta)]\,\phi(\eta)\,d\eta. \quad (2.13)$$

Transforming to the new variable τ we have

$$f'(\eta) = \int_0^\tau e^{-\tau}\phi(\eta)\frac{d\eta}{d\tau}\,d\tau. \quad (2.22)$$

and if we now let

$$\phi(\eta) = \sum_{n=0}^{\infty}b_n\eta^n, \text{ and } \phi(\eta)\frac{d\eta}{d\tau} = \tau^{-2/3}\sum_{m=0}^{\infty}d_m\tau^{m/3} \quad (2.23)$$

where the expansion starts with $\tau^{-2/3}$ because η starts with $\tau^{1/3}$, we find, by a procedure similar to that above, that

$$d_m = \frac{1}{6\pi i}\oint_3\phi(\eta)\frac{d\eta}{d\tau}\tau^{-1/3\,(m-1)}\,d\tau = \frac{1}{6\pi i}\oint_1\phi(\eta)\,\tau^{-1/3\,(m+1)}\,d\eta \quad (2.24)$$

where the first integral is taken three times round a circuit in the τ-plane containing $\tau = 0$, and the second integral is taken once round a circuit in the η-plane containing $\eta = 0$. Hence it follows that d_m is equal to 1/3 of the coefficient of η^{-1} in the expansion in ascending powers of η of the expression $\phi(\eta)\,\tau^{-1/3\,(m+1)}$, i.e. 1/3 of the coefficient of η^m in the expression

$$(c_0 + c_1\eta + c_2\eta^2 + \ldots)^{-1/3\,(m+1)}(b_0 + b_1\eta + b_2\eta^2 + \ldots).$$

The coefficients b_m are most easily found by using (2.12), viz. $f''(\eta) = e^{-\tau}\phi(\eta)$, and substituting for f'', $e^{-\tau}$ and $\phi(\eta)$ from (2.9), (2.15), and

(2.23) respectively. Then the equating of coefficients of powers of η gives the b_m in terms of a_2 and a_4. In fact the substitution gives

$$\left(2a_2 - \eta + 12a_4\eta^2 + \frac{2}{3}a_2^2\eta^3 + \ldots\right)\left(1 + \tau + \frac{\tau^2}{2!} + \frac{\tau^3}{3!} + \ldots\right) = b_o + b_1\eta + b_2\eta^2 + \ldots \quad (2.25)$$

where

$$\tau = a_2\eta^3 - \frac{1}{8}\eta^4 + \frac{3}{5}a_4\eta^5 + \frac{1}{60}a_2^2\eta^6 + \ldots \quad (2.15)$$

Thus

$$b_o = 2a_2, b_1 = -1, \quad b_2 = 12a_4, \quad b_3 = \frac{8}{3}a_2^2,$$
$$b_4 = -\frac{5}{4}a_2, \text{ etc.}$$

and d_m is 1/3 of the coefficient of η^m in the expression

$$\left(a_2 - \frac{1}{8}\eta + \frac{3}{5}a_4\eta^2 + \ldots\right)^{-1/3\,(m+1)}(2a_2 - \eta + 12a_4\eta^2 + \ldots) \quad (2.26)$$

From (2.23) and (2.24) we obtain

$$\int_0^\infty e^{-\tau}\phi(\eta)\,d\eta = \sum_{m=0}^{\infty} d_m\,\Gamma\left(\frac{m+1}{3}\right) \quad (2.27)$$

and to find the value of the integral from 0 to τ we use the incomplete gamma function.

The actual values of the first few coefficients of the A_m and d_m series of (2.17) and (2.23) are as follows.

$$\left.\begin{aligned} A_0 &= a_2^{-1/3}, \quad A_1 = \frac{1}{12}a_2^{-5/3}, \\ A_2 &= a_2^{-1}\left(-\frac{3}{5}\frac{a_4}{a_2} + \frac{1}{64a_2^2}\right) \\ A_3 &= a_2^{-4/3}\left(-\frac{a_2}{45} - \frac{7a_4}{30a_2^2} + \frac{35}{81\cdot128a_2^3}\right) \\ A_4 &= a_2^{-5/3}\left[-\frac{1}{108} + \frac{8}{5}\frac{a_4^2}{a_2^2} - \frac{11}{144}\frac{a_4}{a_2^3} + \frac{6160}{1944}\frac{1}{(8a_2)^4}\right] \text{ etc.} \end{aligned}\right\} \quad (2.28)$$

$$\left.\begin{aligned} d_o &= \frac{2}{3}a_2^{2/3}, \quad d_1 = -\frac{5}{18}a_2^{-2/3}, \\ d_2 &= a_2^{-1}\left(\frac{5}{18}a_4 - \frac{1}{32a_4}\right) \\ d_3 &= \frac{1}{3}a_2^{-4/3}\left[\frac{79}{30}a_2^2 + \frac{77}{30}\frac{a_4}{a_2} - \frac{91}{81}\frac{1}{(8a_2)^2}\right] \end{aligned}\right\} \quad (2.28)$$

$$d_4 = \frac{1}{3}a_2^{-5/3}\left[-\frac{19}{54}a_2 - \frac{127}{15}\frac{a_4^2}{a^2} + \frac{49}{72}\frac{a_4}{a_2^2} - \frac{55}{486}\frac{1}{(8a_2)^3}\right]. \quad (2.29)$$

The integral

$$F = 1 + D\int \exp\left[-3\sigma\int\omega(\eta)\right] \quad (2.13)$$

may be evaluated by a similar procedure, leading to a second equation, from the boundary condition on F at ∞, for the unknown parameters a_2 and a_4. Transforming to the new variable τ, as above, we have

$$F(\eta) = 1 + D\int e^{-\sigma\tau}\frac{d\eta}{d\tau}\,d\tau, \quad (2.30)$$

whence, writing

$$\eta = \sum_{m=0}^{\infty}\frac{A_m}{m+1}\tau^{1/3\,(m+1)},$$

where the A_m are given in (2.28) above, we have

$$F(\infty) = 1 + D\sum_0^\infty \frac{1}{3}A_m\,\Gamma\left(\frac{m+1}{3}\right)\sigma^{-1/3\,(m+1)}. \quad (2.31)$$

The value of D is obtainable from the condition that $F' = -24a_4$ at $\eta = 0$, hence $D = -24a_4$, and the value of F at an interior point of the range is given by the incomplete gamma function.

The conditions $F = 0 = f'$ at $\eta = \infty$ now give us a pair of algebraic equations from which to determine a_2 and a_4. In the case where the series (2.27) and (2.31) are divergent we may, as Meksyn [14] has pointed out, make them formally convergent by introducing an arbitrary small parameter ϵ, and writing in (2.23) and (2.30) $\exp(-\tau\epsilon^{-3})$ and $\exp(-\sigma\tau\epsilon^{-3})$, instead of $e^{-\tau}$ and $e^{-\sigma\tau}$ respectively. The expansions (2.27) and

(2.31) are then divergent for $\epsilon = 1$, but can be summed by applying Euler's transformation to the parameter ϵ.

3. COMBINED FREE AND FORCED CONVECTION: CONSTANT WALL TEMPERATURE

We next consider the problem of free convection at a vertical wall in the presence of a vertical mainstream velocity; the difference between wall temperature and temperature of the ambient fluid being constant, and the mainstream velocity varying as $x^{1/2}$ in accordance with the conditions of Section 1. Writing $\Delta T(x) = T_1 - T_o$, and supposing

$$U_o = Ux^{1/2} \tag{3.1}$$

where T_1 is the wall temperature and U a constant, we make the transformations

$$\xi = x, \quad \eta = \frac{1}{2}\left(\frac{U}{\nu}\right)^{1/2}\frac{y}{x^{1/4}}, \quad f(\xi,\eta) = \frac{\psi(x,y)}{2(U\nu)^{1/2}x^{3/4}}$$

$$F(\xi,\eta) = \frac{T(x,y) - T_o}{T_1 - T_o}. \tag{3.2}$$

The assumption that f and F are functions of η only then leads us to the equations

$$f''' + 3ff'' - 2f'^2 + 2 + \frac{4g\beta(T_1 - T_o)}{U^2}F = 0 \tag{3.3}$$

$$F'' + 3\sigma fF' = 0. \tag{3.4}$$

The coefficient of F in equation (3.3), which measures the relative importance of the free convection, is effectively the ratio of a Grashof number to square of Reynolds number.

In finding solutions of the equations (3.3) and (3.4) we follow fairly closely the working of Section 2. Assuming a series solution for f and F, and applying the boundary conditions at $\eta = 0$; i.e. $f = f' = 0$, $F = 1$; we find

$$f = \sum_2^\infty a_r \eta^r$$

$$F = 1 + \sum_2^\infty \alpha_r \eta^r. \tag{3.5}$$

Substituting the expressions (3.5) into equations (3.3) and (3.4) we have

$$\sum_3^\infty a_r r(r-1)(r-2)\eta^{r-3} + 3\sum_2^\infty a_r\eta^r \sum a_r r(r-1)\eta^{r-2} - 2\Big(\sum_2^\infty a_r r\eta^{r-1}\Big)^2 + (2+\lambda) + \lambda\sum_1^\infty \alpha_r\eta^r = 0 \tag{3.6}$$

$$\sum_2^\infty \alpha_r r(r-1)\eta^{r-2} + 3\sigma\sum_2^\infty a_r\eta^r \sum_1^\infty \alpha_r r\eta^{r-1} = 0 \tag{3.7}$$

writing

$$\lambda = \frac{4g\beta(T_1 - T_o)}{U^2} \tag{3.8}$$

and equating to zero the coefficients of various powers of η in (3.6) and (3.7)

$$\eta^0\begin{cases} 6a_3 + 2 + \lambda = 0, & a_3 = -\dfrac{2+\lambda}{6} \\ 2\alpha_2 = 0, & \alpha_2 = 0 \end{cases}$$

$$\eta^1\begin{cases} 24a_4 + \lambda\alpha_1 = 0, & \alpha_1 = -\dfrac{24a_4}{\lambda} \\ 6\alpha_3 = 0, & \alpha_3 = 0 \end{cases}$$

$$\eta^2\begin{cases} 60a_5 + 6a_2^2 - 8a_2^2 + \lambda\alpha_2 = 0, & a_5 = \dfrac{1}{30}a_2^2 \\ 12\alpha_4 + 3\sigma a_2\alpha_1 = 0, & \alpha_4 = -\dfrac{1}{4\sigma}a_2\alpha_1 \\ & \quad = \dfrac{6\sigma a_2 a_4}{\lambda} \end{cases}$$

$$\eta^3\begin{cases} 120a_6 + 18a_3a_2 + 6a_3a_2 - 24a_3a_2 + \lambda\alpha_3 = 0, \\ \qquad a_6 = 0 \\ 20\alpha_5 + 3\sigma(2a_2\alpha_2 + a_3\alpha_1) = 0, \\ \qquad \alpha_5 = -\dfrac{3\sigma a_3\alpha_1}{20} \\ \qquad\quad = -\dfrac{3(2+\lambda)\sigma a_4}{5\lambda} \end{cases}$$

$$\eta^4\begin{cases} 210a_7 + 10a_2a_4 + \lambda\alpha_4 = 0 \\ 30\alpha_6 + 3\sigma(3a_2\alpha_3 + 2a_3\alpha_2 + a_4\alpha_1) = 0 \end{cases}$$

$$\therefore a_7 = -a_2a_4\left(\frac{1}{21} + \frac{\sigma}{35\lambda}\right),$$

$$\alpha_6 = -\frac{\sigma}{10}a_4\alpha_1 = \frac{12\sigma}{5\lambda}a_4^2, \text{ etc.}$$

Thus we can write

$$\left.\begin{aligned} f' &= a_2\eta^2 - \frac{2+\lambda}{6}\eta^3 + a_4\eta^4 + \frac{1}{30}a_2^2\eta^5 \\ &\quad - a_2a_4\left(\frac{1}{21}+\frac{\sigma}{35\lambda}\right)\eta^7 + \ldots \\ F &= 1 - \frac{24a_4}{\lambda}\eta + \frac{6\sigma a_2 a_4}{\lambda}\eta^4 \\ &\quad - \frac{3(2+\lambda)}{5\lambda}\sigma a_4\eta^5 + \frac{12\sigma}{5\lambda}a_4^2\eta^6 + \ldots \end{aligned}\right\} \quad (3.9)$$

whence, substituting in (3.3) and (3.4) for f, f' and F, and writing $\tilde{\omega}$ and ρ respectively for the series representations (2.5) of f' and F, we find

$$f''' + 3\tilde{\omega}f'' = 2\tilde{\omega}'^2 - 2 - \lambda\rho \quad (3.10)$$

$$F'' + 3\sigma\tilde{\omega}F' = 0 \quad (3.11)$$

which may be integrated to give

$$f'' = \exp[-3\textstyle\int\tilde{\omega}(\eta)]\,\{\int_0^\eta \exp[3\int\tilde{\omega}(\eta)]\,(2\tilde{\omega}'^2 - 2 - \lambda\rho)\,d\eta + C\}$$

$$F' = D\exp[-3\sigma\textstyle\int\tilde{\omega}(\eta)] \quad (3.12)$$

where C and D are constants.

To complete the solution we must now evaluate the integrals

$$f' = \textstyle\int \exp[-3\int\tilde{\omega}(\eta)\,d\eta]\,\phi(\eta)\,d\eta,$$

$$F = 1 + D\textstyle\int\exp[-3\sigma\int\tilde{\omega}(\eta)\,d\eta]\,d\eta \quad (3.13)$$

where we have written

$$\phi(\eta) = \textstyle\int \exp[3\int\tilde{\omega}(\eta)\,d\eta]\,(2\tilde{\omega}'^2 - 2 - \lambda\rho)\,d\eta + C. \quad (3.14)$$

The series representation $\tilde{\omega}(\eta)$ for f' given by (3.9) may be integrated term by term to give

$$3\textstyle\int\tilde{\omega}(\eta)\,d\eta = a_2\eta^3 - \frac{2+\lambda}{8}\eta^4 + \frac{3}{5}a_4\eta^5 + \frac{1}{60}a_2^2\eta^6 - \frac{3}{8}a_2a_4\left(\frac{1}{21}+\frac{\sigma}{35\lambda}\right)\eta^8 + \ldots = \tau, \text{ say} \quad (3.15)$$

and it is now possible to replace η by τ as the independent variable in the expression (3.13) for f', viz.

$$f' = e^{-\tau}\phi(\eta)\,d\eta = \textstyle\int e^{-\tau}\phi(\eta)\frac{d\eta}{d\tau}\,d\tau. \quad (3.16)$$

For small values of τ we can express η as a power series in τ,

$$\eta = \sum_{m=0}^{\infty}\frac{A_m}{m+1}\tau^{1/3(m+1)} \quad (3.17)$$

where the coefficient A_m is the coefficient of η^m in the expression

$$\left[a_2 - \frac{2+\lambda}{8}\eta + \frac{3}{5}a_4\eta^2 + \frac{1}{60}a_2^2\eta^2 - \frac{3}{8}a_2a_4\left(\frac{1}{21}+\frac{\sigma}{35\lambda}\right)\eta^5 + \ldots\right]^{1/3(m+1)}.$$

Moreover, if

$$\phi(\eta) = \sum_{n=0}^{\infty} b_n\eta^n, \quad \text{and} \quad \phi(\eta) = \tau^{-2/3}\sum_{m=0}^{\infty} d_m\tau^{m/3} \quad (3.18)$$

it follows, as in Section 2, that the coefficient d_m is equal to 1/3 of the coefficient of η^m in the expression

$$\left(a_2 - \frac{2+\lambda}{8}\eta + \frac{3}{5}a_4\eta^2 + \frac{1}{60}a_2^2\eta^3 + \ldots\right)^{-1/3(m+1)}(b_o + b_1\eta + b_2\eta^2 + \ldots)$$

and we can thus express the integral (2.16) in the form of a series of gamma functions.

The coefficients b_r are obtained from the equation

$$f'' = e^{-\tau}\phi(\eta) \quad (3.12)$$

i.e. $b_o + b_1\eta + \ldots = \tilde{\omega}''e^{\tau}$

$$= \left[2a_2 - (2+\lambda)\eta + 12a_4\eta^2 + \frac{2}{3}a_2^2\eta^3 - 6a_2a_4\left(\frac{1}{3}+\frac{\sigma}{5\lambda}\right)\eta^5 + \ldots\right]e^{\tau}.$$

and

$$e^{\tau} = 1 + a_2\eta^3 - \left(\frac{2+\lambda}{8}\right)\eta^4 + \frac{3}{5}a_4\eta^5 + \frac{31}{60}a_2^2\eta^6 + \ldots,$$

so it follows that

$$b_0 = 2a_2, \quad b_1 = -(2+\lambda), \quad b_2 = -12a_4,$$

$$b_3 = \frac{8}{3}a_2^2, \quad b_4 = -\frac{5}{4}(2+\lambda)a_2,$$

$$b_5 = \frac{a_2 a_4}{5}\left(56 - \frac{6\sigma}{5\lambda}\right) - \frac{(2+\lambda)^2}{8}, \text{ etc.} \tag{3.19}$$

The coefficients A_m and d_m of (3.17) and (3.18) may now be written down as follows,

$$\left.\begin{aligned} A_0 &= a_2^{-1/3}, \quad A_1 = \frac{1}{12}(2+\lambda)a_2^{-5/3}, \\ A_2 &= a_2^{-1}\left[-\frac{3}{5}\frac{a_4}{a_2} + \frac{(2+\lambda)^2}{64a_2^2}\right], \\ A_3 &= a_2^{-4/3}\left[-\frac{a_2}{45} - \frac{7(2+\lambda)}{30}\frac{a_4}{a_2^2} + \frac{35(2+\lambda)^2}{81\cdot 128a_2^3}\right], \text{ etc.} \end{aligned}\right\} \tag{3.20}$$

$$\left.\begin{aligned} d_0 &= \frac{2}{3}a_2^{2/3}, \quad d_1 = -\frac{5}{18}(2+\lambda)a_2^{-2/3}, \\ d_2 &= a_2^{-1}\left[\frac{18}{5}a_4 - \frac{(2+\lambda)^2}{32a_2}\right], \\ d_3 &= \frac{1}{3}a_2^{-4/3}\left[\frac{118}{45}a_2^2 + \frac{7(2+\lambda)}{3}\frac{a_4}{a_2} - \frac{91}{81}\frac{(2+\lambda)^3}{(3a_2)^2}\right], \text{ etc.} \end{aligned}\right\} \tag{3.21}$$

The integral (3.16) may now be expressed as a series of gamma functions as follows.

$$f'(\eta) = \int e^{-\tau}\phi(\eta)\frac{d\eta}{d\tau}d\tau = \sum_{m=0}^{\infty} d_m \Gamma_\tau\left(\frac{m+1}{3}\right) \tag{3.22}$$

and in the case of the second integral of (3.13), namely

$$F = 1 + D\int \exp\left[-3\sigma\int\bar{\omega}(\eta)\right]d\eta,$$

we again change the variable to τ, and write

$$F = 1 + D\int e^{-\sigma\tau}\frac{d\eta}{d\tau}d\tau \tag{3.23}$$

whence, writing

$$\eta = \sum_{m=0}^{\infty}\frac{A_m}{m+1}\tau^{-1/3(m+1)},$$

where the A_m are given in (3.20) above, we have

$$F(\eta) = 1 + \frac{1}{3}D\sum_{m=0}^{\infty}A_m\sigma^{-1/3(m+1)}\Gamma_{\sigma\tau}\left(\frac{m+1}{3}\right). \tag{3.24}$$

The condition on $F'(\eta)$ at $\eta = 0$ deducible from equation (3.9), namely $F'(0) = -24a_4/\lambda$, means that we have $D = -24a_4/\lambda$; the conditions $F = 0$, $f' = 1$ at $\eta = \infty$ then provide two equations from which to determine a_2 and a_4.

In the case in which λ is large, which means that the natural convection is more important than the forced convection, it is convenient to use a different transformation of variables. Instead of (3.2) we write

$$\left.\begin{aligned} \xi &= x, \quad \eta = \left[\frac{g\beta(T_1 - T_0)}{4\nu^2}\right]^{1/4}\frac{y}{x^{1/4}} \\ f(\xi,\eta) &= \frac{\psi(x,y)}{4\nu\left[\frac{g\beta(T_1-T_0)}{4\nu^2}\right]^{1/4}x^{3/4}} \\ F(\xi,\eta) &= \frac{T(x,y) - T_0}{T_1 - T_0} \end{aligned}\right\} \tag{3.25}$$

and in this case the equations of momentum and energy become

$$f''' + 3ff'' - 2f'^2 + \frac{U^2}{2g\beta(T_1 - T_0)} + F = 0 \tag{3.26}$$

$$F'' + 3\sigma fF' = 0. \tag{3.27}$$

Thus if we write

$$\mu = \frac{U^2}{g\beta(T_1 - T_0)} \tag{3.28}$$

so that $\mu = 4/\lambda$, then all the analysis above applies, provided that we write $1/2\,\mu + 1$ for $2 + \lambda$, and remove the factor $1/\lambda$ from the a_4.

However the boundary condition at ∞ on f is now changed somewhat, for we have, at ∞, $\partial\psi/\partial y = Ux^{1/2}$.

Thus

$$4\nu\left[\frac{g\beta(T_1 - T_0)}{4\nu^2}\right]^{1/4}x^{3/4}\frac{\partial f}{\partial\eta}\left[\frac{g\beta(T_1 - T_0)}{4\nu^2}\right]^{1/4}x^{-1/4} = Ux^{1/2}$$

i.e.

$$2\,[g\beta(T_1 - T_o)]^{1/2}\,x^{1/2} f' = U x^{1/2} \qquad (3.29)$$

and hence

$$f' = \frac{U}{2[g\beta(T_1 - T_o)]^{1/2}} = 1/2\sqrt{\mu}. \qquad (3.30)$$

The condition at ∞ on F is of course $F = 0$ as above.

4. COMBINED FREE AND FORCED TRANSFER: UNIFORM HEAT FLUX

The rate at which heat is transmitted from the wall to the fluid is, according to Fourier's law, proportional to $\left(\frac{\partial T}{\partial y}\right)_{y=0}$. By uniform heat flux we mean that this rate of heat transfer is independent of x, and in order that this should be so we require that $\Delta T(x) \propto x^{1/5}$ and $U_o \propto x^{3/5}$. Again, as in Section 3, we choose two separate similarity transformations, according as the forced convection or the free convection is regarded as dominant. In the former case we write

$$\xi = x, \quad \eta = \left(\frac{U}{5\nu}\right)^{1/2} \frac{y}{x^{1/5}}, \quad f(\xi, \eta) = \frac{\psi(x, y)}{(5\nu U)^{1/2} x^{4/5}}$$

$$F(\xi, \eta) = \frac{T(x, y) - T_o}{T_1 - T_o} \qquad (4.1)$$

which gives equations

$$f''' + 4ff'' - 3f'^2 + \frac{5g\beta(T_1 - T_o)}{U^2} F + 3 = 0 \qquad (4.2)$$

$$F'' + \sigma(4fF' - f'F) = 0. \qquad (4.3)$$

We proceed with the solutions of equations (4.2) and (4.3) in the manner developed in Section 2, and eventually express both f' and F as a series of gamma functions, viz.

$$f' = \sum_{m=0}^{\infty} d_m\, \Gamma_\tau\left(\frac{m+1}{3}\right) \qquad (4.4)$$

$$F = 1 + \sum_{m=0}^{\infty} A_m\, \Gamma_{\sigma\tau}\left(\frac{m+1}{3}\right) \sigma^{-1/3(m+1)} \qquad (4.5)$$

where $\tau(\eta)$ is defined in a similar manner to that in Section 3.

In this case the d_m and the A_m are as follows.

$$\left.\begin{aligned}
A_o &= \left(\frac{4}{3}a_2\right)^{-1/3}, \quad A_1 = \left(\frac{4}{3}a_2\right)^{-2/3} \frac{3+\lambda}{12a_2}, \\
A_2 &= \left(\frac{4}{3}a_2\right)^{-1}\left[-\frac{3a_4}{5a_2} + \frac{(3+\lambda)^2}{64a_2^2}\right] \\
A_3 &= \left(\frac{4}{3}a_2\right)^{-4/3}\left[-\frac{2}{45}a_2 - \frac{7(3+\lambda)}{30}\frac{a_4}{a_2^2} + \frac{35(3+\lambda)^2}{81\cdot 128 a_2^3}\right]
\end{aligned}\right\} \qquad (4.6)$$

$$\left.\begin{aligned}
d_o &= \frac{2}{3}a_2\left(\frac{4}{3}a_2\right)^{-1/3} \\
d_1 &= -\frac{5}{18}(3+\lambda)\left(\frac{4}{3}a_2\right)^{-2/3}, \\
d_2 &= \left(\frac{4}{3}a_2\right)^{-1}\left[\frac{18}{5}a_4 - \frac{(3+\lambda)^2}{32a_2}\right], \\
d_3 &= \left(\frac{4}{3}a_2\right)^{-4/3}\left[\frac{56}{45}a_2^2 + \frac{7}{3}(3+\lambda)\frac{a_4}{a_2} - \frac{91}{81}\frac{(3+\lambda)^3}{(8a_2)^2}\right]
\end{aligned}\right\} \qquad (4.7)$$

where

$$\lambda = \frac{5g\beta(T_1 - T_o)}{U^2},$$

and the conditions $F(\infty) = 0, f'(\infty) = 1$ provide two equations for a_2 and a_4.

If, on the other hand, the free convection is to be regarded as dominant, an appropriate transformation of variables is

$$\left.\begin{aligned}
\xi &= x, \quad \eta = \left[\frac{g\beta(T_1 - T_o)}{5\nu^2}\right]^{1/4} \frac{y}{x^{1/5}} \\
f(\xi, \eta) &= \frac{\psi(x, y)x^{-4/5}}{5\nu\left[\frac{g\beta(T_1 - T_o)}{5\nu^2}\right]^{1/4}} \\
F(\xi, \eta) &= \frac{T(x, y) - T_o}{T_1 - T_o}.
\end{aligned}\right\} \qquad (4.8)$$

5. RESULTS AND DISCUSSIONS

Some numerical results based on the analysis of Sections 2, 3 and 4 have been computed on the University of Leeds Pegasus computer, and a selection of these results is presented here. In all cases the computations have been based on the first three terms of the appropriate gamma function expansion for the temperature and stream function. Numerical results for the case of free convection at a vertical plate at uniform temperature are summarized in Figs. 1 and 2, and a direct comparison between the results obtained above for $\sigma = 0{\cdot}733$, and those obtained by Schmidt and Beckmann [1], Saunders [3] and Squire [16], illustrated in Fig. 3, shows an agreement good enough to provide confidence in the accuracy of the approximation (at least for values of σ not too small compared with unity).

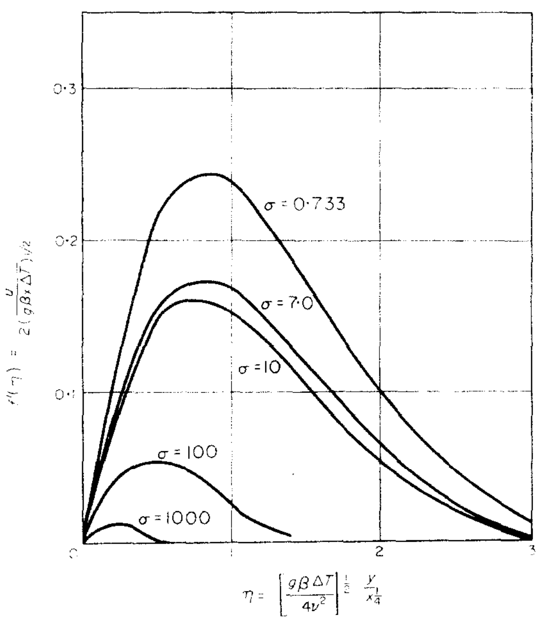

FIG. 1. Representative velocity profiles for varying values of σ in the pure free convection case.

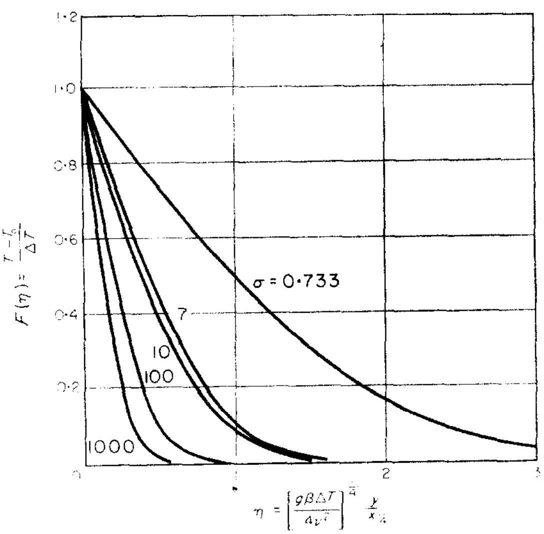

FIG. 2. Representative temperature profiles for varying values of σ in the pure free convection case.

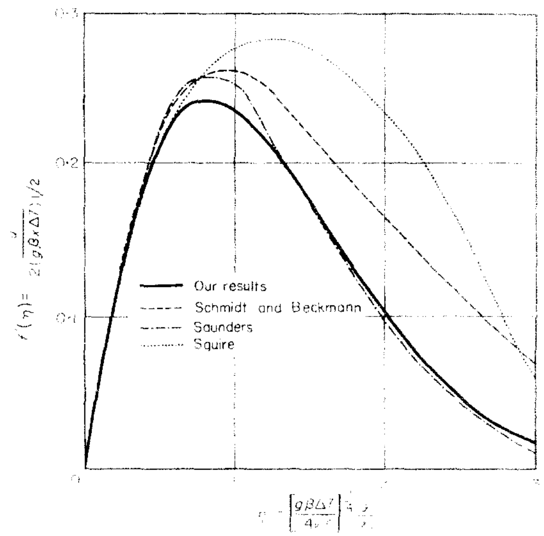

FIG. 3. Comparison of the velocity profile obtained in this paper with those obtained by earlier workers ($\sigma = 0{\cdot}733$).

The rate of heat transfer from the plate is given by

$$q = -k\left(\frac{\partial T}{\partial y}\right)_{y=0},$$

which, using equations (2.1) and the numercial result obtained for $F'(0)$ is, for $\sigma = 0{\cdot}733$,

$$q = -0{\cdot}529k\,\Delta T\left(\frac{g\beta\Delta T}{4\nu^2 x}\right)^{1/4},$$

and hence the mean Nusselt number for a plate of height l is

$$Nu = 0{\cdot}499\left(\frac{g\beta l^3\Delta T}{\nu^2}\right)^{1/4};$$

this value differs from that obtained numerically by Ostrach [2] by a margin of about 4 per cent. and in Table 1 is presented a comparison between our results and those of Ostrach for various values of σ.

Table 1. Values of temperature gradient and Nusselt number at the vertical wall, for the case of constant temperature

σ	0·01	0·733	7	10	100	1000
Our—$F'(0)$	0·051	0·529	1·168	1·300	2·346	3·941
Ostrach's—$H'(0)$	0·081	0·508	—	1·169	2·191	3·966
$Nu \Big/ \left[\frac{g\beta l^3(T_1 - T_o)}{\nu^2}\right]^{1/4}$	0·485	0·499	1·100	1·227	2·200	3·60

The measure of agreement obtained between the results presented here, and numerical results obtained by other investigators suggests that this method of approximation is a very useful one in free convection problems. In our case, where three terms of the series expansions are used, the essential computational problem consists of finding the smallest solution of a quantic equation in which the Prandtl number appears as a parameter. It is thus a matter of little labour to find solutions for any σ, but it should be noted that the method does not work at all well for very small Prandtl number because of the presence of factors of the form $\sigma^{-1/3(m+1)}$ (m an integer) in the expansions for the temperature field. Physically this means that the temperature boundary layer is much thicker than the velocity boundary layer, and so the polynomial expression for the velocity is used in the equations over a much greater range of η than that for which it is an accurate approximation.

Numerical results for the case of combined free and forced convection at a vertical wall kept at uniform temperature are summarized in Figs. 4 and 5. Again the essential computational problem is the finding of the first root of a quartic equation, which contains as parameters the Prandtl number σ and the quantity μ (or λ), which is effectively of the form Re^2/Gr, where Re is the Reynolds number, and Gr the Grashof number appropriate to the problem.

The shear stress at the wall is

$$\frac{\nu}{\rho}\left(\frac{\partial u}{\partial y}\right)_{y=0},$$

and the heat transfer

$$k\left(\frac{\partial T}{\partial y}\right)_{y=0},$$

it is convenient to measure these in terms of coefficients defined as follows, the Nusselt number

$$Nu = \frac{x\left(\frac{\partial T}{\partial y}\right)_{y=0}}{T_1 - T_o}$$

and the friction coefficient

$$c_f = \frac{\frac{\nu}{\rho}\left(\frac{\partial u}{\partial y}\right)_{y=0}}{1/2\rho U_\infty^2}.$$

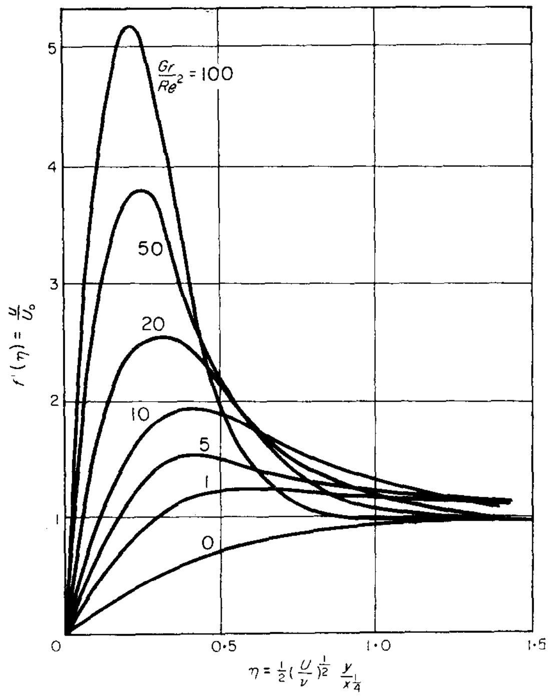

FIG. 4. Representative velocity profiles for the uniform wall temperature case ($\sigma = 0·733$).

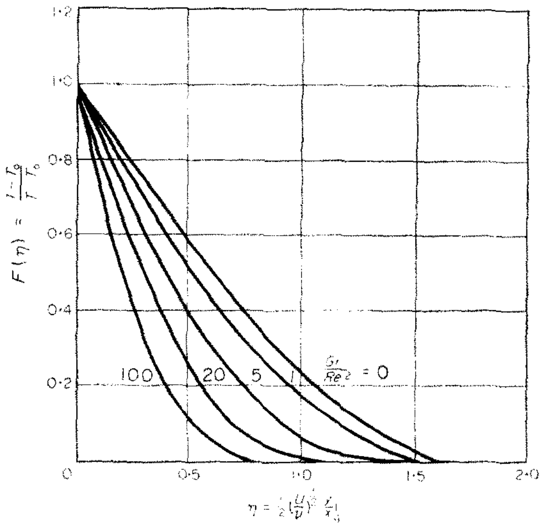

FIG. 5. Representative temperature profiles for the uniform wall temperature case ($\sigma = 0{\cdot}733$).

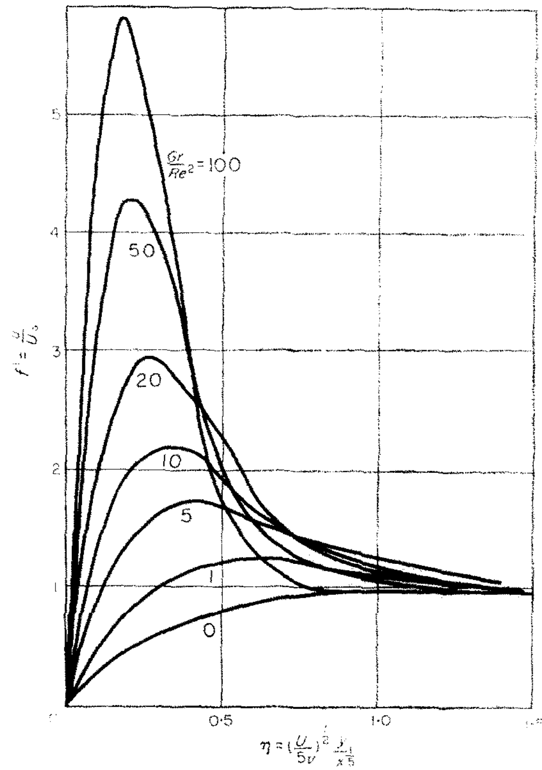

FIG. 6. Representative velocity profiles for the uniform heat flux case ($\sigma = 0{\cdot}733$).

Variations of these coefficients for varying Gr/Re^2 are shown in Table 2, all the figures being based on a Prandtl number of 0·733. In order to facilitate comparison with the results of Sparrow, Eichhorn and Gregg [13], the quantities tabulated are $2c_f Re^{1/2}$ and $2Nu Re^{-1/2}$. Agreement is on the whole very good, but our values of heat transfer are systematically higher by some 7 per cent, and for small values of Gr/Re^2 the difference in the skin friction figures runs to 12–14 per cent.

Some results have also been prepared for the uniform heat flux case, this time for both $\sigma = 0{\cdot}733$ and $\sigma = 7$. These results are presented in Figs. 6 and 7 and Table 3, and again agreement with Sparrow, Eichhorn and Gregg is very good where a direct comparison is possible.

In conclusion, it seems apparent that the symptotic methods developed by Meksyn and used by him in a variety of boundary-layer

Table 2. Heat transfer and shear stress coefficients for case of constant wall temperature ($\sigma = 0{\cdot}733$)

Gr/Re^2	100	50	20	10	5	2	1	0·25	0·125	0
$2c_f Re^{1/2}$	120·76	74·70	37·56	23·88	15·41	9·330	6·942	4·928	4·564	4·196
$2Nu Re^{-1/2}$	2·433	2·068	1·685	1·458	1·282	1·115	1·042	0·953	0·937	0·921

Table 3. Heat transfer and shear stress coefficients for case of uniform heat flux [(a) $\sigma = 0{\cdot}733$, (b) $\sigma = 7$]

Gr/Re^2		100	50	20	10	5	1	0
$2c_f Re^{1/2}$	(a)	119·02	72·10	37·76	23·78	15·45	7·216	4·592
	(b)	93·80	56·88	29·94	18·92	12·45	6·264	4·592
$2Nu Re^{-1/2}$	(a)	2·474	2·106	1·720	1·491	1·314	1·070	0·965
	(b)	5·412	4·598	3·739	3·234	2·846	2·340	2·177

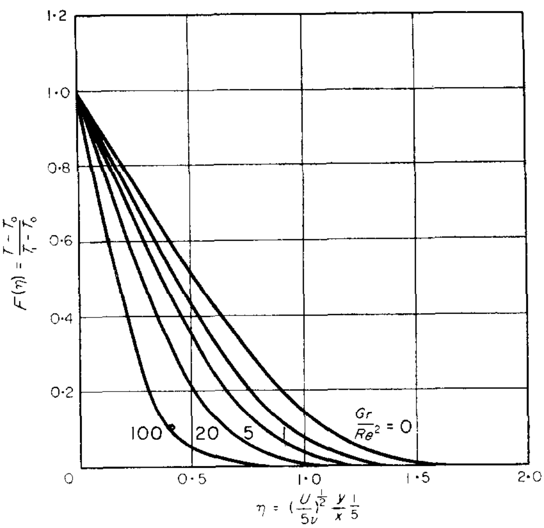

FIG. 7. Representative temperature profiles for the uniform heat flux case ($\sigma = 0{\cdot}733$).

problems may be usefully applied to a wide range of problems in both free and combined free and forced convection. Only a small number of terms need be retained in the asymptotic series in order to give a good approximation to the accurate solutions as derived by purely numerical methods of the equations of motion and energy, and the effect of variation in the important parameters of the problems, for example Prandtl number, or the quantity G_r/Re^2, are more clearly seen and more easily dealt with than in the direct numerical approach. The limitations of the method lie of course in its requirement of similarity solutions, and its increasing inaccuracy for decreasing Prandtl number, as mentioned in Section 1, but nevertheless its application in free convection problems requiring theoretical solutions of a moderate degree of accuracy, say error < 10 per cent, could be wide.

ACKNOWLEDGEMENT

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Résumé—Une technique d'approximation largement utilisée par Meksyn pour établir des solutions en développement asymptotique aux problèmes de couche limite est étudiée ici, en écoulement de convection naturelle et appliquée au problème classique de la convection naturelle sur une plaque verticale uniformément chauffée dans un fluide au repos à l'infini, ainsi qu'au problème moins classique de convection naturelle combinée à la convection forcée sur une plaque verticale dans un fluide ayant une vitesse verticale au loin de la plaque. On trouve que dans le fluide considéré les trois premiers termes de la série asymptotique fournissent une bonne approximation pour les résultats connus et puisque dans ce cas le problème de calcul essentiel est de trouver les plus petites racines d'une équationd u 4° ordre dans laquelle le nombre de Prandtl apparaît comme paramètre la méthode est bien adaptée et d'application plus générale que celle utilisée par d'autres auteurs dans ces problèmes.

D'autres questions de convection naturelle ou de combinaison convection naturelle et forcée dans lesquelles des transformations semblables peuvent être utilisées sont justiciables de la même technique.

Zusammenfassung—Ein von Meksyn verwendetes Näherungsverfahren zur Ermittlung von Lösungen für Strömungsprobleme in Grenzschichten in Form asymptotischer Erweiterungen wird auf freie Konvektionsströme ausgedehnt und auf das klassische Problem der freien Konvektion an einer gleichmässig beheizten senkrechten Wand in sonst ruhendem Medium angewandt. Daneben werden auch die weniger bekannten Probleme der kombinierten freien und erzwungenen Konvektion an einer senkrechten Wand in einem Medium mit Vertikalgeschwindigkeit in grossem Abstand von der Wand behandelt. In den erwähnten Fällen zeigt sich, dass die ersten drei Glieder der asymptotischen Reihe eine gute Näherung für bekannte Resultate darstellen. Da hierbei das wesentliche Berechnungsproblem im Auffinden der kleinsten Wurzel einer Reihengleichung liegt, in der die Prandtlzahl als Parameter vorkommt, ist die Methode leichter zu handhaben und von allgemeinerer Verwendbarkeit als jene Verfahren früherer Bearbeiter dieser Probleme. Andere Erscheinungen der freien Konvektion oder der kombinierten freien und erzwungenen Konvektion, in welcher Ähnlichkeitstransformationen verwendet werden können, sind unmittelbar der Methode zugänglich.

Аннотация—Приближенная методика, широко применяемая Мексиным для нахождения решения задач о течении в пограничных слоях в виде асимптотических разложений, распространена на течения при свободной конвекции и применена в классических задачах о свободной конвекции у равномерно нагреваемой вертикальной стенки в жидкости, а также в других менее известных задачах о совместной свободной конвекции у вертикальной стенки в жидкости, имеющей вертикальную скорость на большом расстоянии от стенки. Установлено, что в расематриваемых задачах первые три члена асимптотического ряда дают хорошее приближение к известным результатам, и так как в этом случае задача вычисления заключается в нахождении наименьшего корня чравнения четвертой степени, в котором число Прандтля представлено в виде параметра, этот метод является менее трудоемким и более универсальным, чем методы, применяемые другими авторами для решения подобных задач. Другие задачи о свободной конвекции или о совместной свободной и вынужденной конвекции, в которых могут бытв применены подобные преобразования, поддаются решению по такой же методике.